Computer Science Tripos, Part II: Denotational Semantics
Supervision 1
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1. Are the following binary relations on the given sets præorders, partial orders or total orders?
   (a) $\mathbb{N}, =$
   (b) $\mathbb{R}, \leq$
   (c) $\mathbb{R}, x \preceq y \overset{\text{def}}{\iff} \lfloor x \rfloor \leq \lfloor y \rfloor$
   (d) the set given by the states of a fixed-size game-of-life board, and $x \preceq y \overset{\text{def}}{\iff}$ the game of life will eventually reach state y after starting in state x.
   (e) $\mathbb{C}, z \preceq w \overset{\text{def}}{\iff} \Re(z) \leq \Re(w) \wedge \Im(z) \leq \Im(w)$
2. Which of the following sequences are increasing with respect to their respective posets?
   - $a_n := (1 + {}^1\!/_n)^n$ in $(\mathbb{Q}, \leq)$
   - $b_n := n^{-1}$ in $(\mathbb{Q}, \geq)$
   - $c_n := n$ in $(\mathbb{N}, =)$
   - $d_n := 1$ in $(\mathbb{N}, \leq)$
3. Which of the above sequences have a least upper bound, or supremum, in the given poset?
4. Which of the following are domains?
   (a) $\mathbb{N}, =$
   (b) $\mathbb{R}, \leq$
   (c) $\mathbb{N} \cup \{\infty\}, x \preceq y \overset{\text{def}}{\iff} \exists a \in \mathbb{N} \cup \{\infty\}. x = a \cdot b$
   (d) V (where $G = (V, E)$ is a directed graph) with $u \preceq v \overset{\text{def}}{\iff} u \rightsquigarrow^* v$
5. How does the domain-theoretic notion of continuity relate to the standard analytic notion of continuity?
6. Consider Ω, the 'vertical' natural numbers. Are all monotone endofunctions on Ω continuous?
7. (Exercise Sheet Question 4(ii)). Let $\mathbb{O}$ be the two-element domain given by $\bot \preceq \top$. Show that for any set X, the strict continuous functions $X_\bot \to \mathbb{O}$ are in 1-1-correspondence with the subsets of X, where $X_\bot$ is the flat domain on X.
8. Show that an 'eventually constant' chain has a supremum. Deduce that every finite poset is a cpo. Then deduce that to show that given domains $(D, \preceq), (E, \sqsubseteq)$ (where D is finite) to show that f is continuous it suffices to show that f is monotone[2].
9. Let $(D, \preceq)$ and $(E, \sqsubseteq)$ be domains. We say that a continuous function $f \in D \to E$ is a *continuous isomorphism* if it is bijective and its inverse f^{-1} is also continuous.
   (a) Show that for f to be a conitnuous isomorphism, it suffices for f to be continuous and bijective and for its inverse to be monotone.
   (b) Find an example of a function that is continuous and bijective, but not a continuous isomorphism.
10. Let A, B be two countably infinite, disjoint sets and let c be an element of neither. Furthermore, consider bijections $e_A : A \to \mathbb{N}$ and $e_b : B \to \mathbb{N}$ and write $a_k = e_A^{-1}(k), b_k = e_B^{-1}(k)$.
    Define a binary relation $\preceq \subseteq D \times D$, where $D := A \cup B \cup \{c\}$ as:

$$v \preceq w \overset{\text{def}}{\iff} \begin{cases} w = c & \text{or} \\ w = v \in B & \text{or} \\ e_A(v) \leq e_B(w) & \text{or} \\ e_A(v) \leq e_A(w) \end{cases}$$

[1] the LaTeX for my name is `Daniel S\"a\"aw`.

[2] I realise this may look small and insignificant but when I did this course this question turned into the standard result I used most frequently.

(a) Show that $(D, \preceq)$ is a domain.

(b) Draw a Hasse diagram of this domain.

(c) Let $(E, \sqsubseteq)$ be a domain and let f, g be continuous functions in $D \to E$. Show that $\forall n. f(b_n) = g(b_n) \implies f(c) = g(c)$.

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12. Let Ω be **any set**. See the appendix for the standard definition of a measure space.

    (a) Is a singular σ-algebra Σ over Ω a domain, when ordered by subset inclusion?

    (b) Is the set of σ-algebras over Ω a domain, when ordered by subset inclusion?

    (c) Is a measure $m : \Sigma \to \mathbb{R}_{\geq 0}$ a continuous function, where (Ω, Σ) is a measurable space, Σ is ordered by subset inclusion, and $\mathbb{R}_{\geq 0}$ is ordered by $\leq$?

    (d) Is the set of measures defined on the power set of Ω a domain, when the ordering is given by $f \sqsubseteq_\mu g \iff \forall E \subseteq \Omega. f(E) \leq g(E)$?

    (e) A probability space is just a measure space where the measure $\mathbb{P}$ is defined to have $\mathbb{P}(\Omega) = 1$. Now, consider a slightly modified notion of a probability space given by objects $(\Omega, \Sigma, \mathbb{P}')$, where $(\Omega, \Sigma, \mathbb{P})$ is a valid probability space and $\mathbb{P}'$ is a total function from the power set of Ω to the nonnegative reals, obtained by taking values

    $$\mathbb{P}'(E) = \begin{cases} \mathbb{P}(E) & E \in \Sigma \\ 0 & E \notin \Sigma \end{cases}.$$

    Now, consider the set of objects $(\Omega, \Sigma, \mathbb{P}')$. Order this set as

    $$(\Omega, \Sigma, \mathbb{P}') \preceq (\Omega, \Pi, \mathbb{Q}') \iff \Sigma \subseteq \Pi \wedge \mathbb{P}' \sqsubseteq_\mu \mathbb{Q}'.$$

    Prove that this forms a domain, and provide the form of the lub of any chain in this domain.

Appendix.

A σ-algebra Σ over a set Ω is a collection of subsets of Ω (i.e. $\Sigma \subseteq \mathcal{P}(\Omega)$) that satisfies

(I) $\Omega \in \Sigma$.

(II) Σ is closed under countable union.

(III) Σ is closed under relative complement, that is $\forall E \in \Sigma. \Omega \setminus E \in \Sigma$. So at least $\varnothing \in \Sigma$ by (I).

A *measurable space* (X, Σ) is just a set X equipped with a σ-algebra Σ over it.

A *measure space* (X, Σ, μ) is a measurable space (X, Σ) equipped with a measure; a function $\mu : \Sigma \to \mathbb{R}_{\geq 0}$ satisfying

(I) σ-additivity: If $\forall n \neq m \in \mathbb{N}. E_n \cap E_m = \varnothing \wedge E = \bigcup_{n \in \mathbb{N}} E_n$, then $\mu(E) = \sum_{n \in \mathbb{N}} E_n$.

(II) $\mu(\varnothing) = 0$.