Computer Science Tripos, Part II: Denotational Semantics
Supervision 2
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Constructions on Domains

1. Suppose $(D, \preceq)$ is a domain. Show that $(\preceq, \sqsubseteq)$ is a domain if $(d, e) \sqsubseteq (d', e') \overset{\text{def}}{\iff} d \preceq d' \wedge e \preceq e'$.
2. Suppose X is a set. Show that $(X_\perp, =_\perp)$ is a domain.
3. Suppose X is a set and that $(D, \preceq)$. Show that any monotone function $f : X_\perp \to D$ is continuous
4. Suppose D, E are CPOs.
   (a) Show that the function CPO $D \to E$ as defined in the lecture notes is indeed a CPO.
   (b) Justify the inference rule $\dfrac{f \sqsubseteq_{D\to E} g \qquad x \sqsubseteq_D y}{f(x) \sqsubseteq_E g(y)}$.
   (c) Find a necessary and sufficient condition on D, E for $D \to E$ to be a domain. (And prove that this is necessary and sufficient)
   (d) Let D, E, F be domains. Recall the composition operator $\circ$ which operates on $(E \to F) \times (D \to E)$.
   Show that $\circ \in (E \to F) \times (D \to E) \to (D \to F)$ is well-defined. Then show that it is continuous in each argument, and deduce that it is a continuous function.
5. Suppose X, Y are two sets and consider $X_\perp, Y_\perp$—the flat domains on these sets.
   Show that a function $f : X_\perp \to Y_\perp$ is continuous if and only if at least one of the following conditions holds:
   (I) f is strict: $f(\perp_X) = \perp_Y$.
   (II) f is constant: $\forall x \in X_\perp. f(x) = f(\perp_X)$.
6. Consider a singleton $\{\top\}$ and the flat domain $\{\top\}_\perp$. Let Ω be the vertical natural numbers.
   Show that there is a bijection between $\Omega \to \{\top\}_\perp$ and Ω.
   Prove or disprove: There exists a continuous isomorphism between these two.

Scott Induction

7. 2002, Paper 8, Question 1
8. 2003, Paper 9, Question 14
9. 2004, Paper 9, Question 15
10. Let D, E be two domains. Suppose $f :\to E$ is monotone.
    Show that f is continuous if and only if for any admissible subset $S \subseteq E$, $f^{-1}S \subseteq D$ is admissible.
11. Let $(D, \preceq)$ be some domain. Let $X \subseteq D$. Define the *admissible closure* $\text{Cl}X$ of X as

$$\text{Cl}X \coloneqq \bigcap_{\substack{S \subseteq D:\\ X \subseteq S,\\ S \text{ adm.}}} S$$

    (a) Show that $\text{Cl}X$ is admissible.
    (b) Verify that $\text{Cl}X$ is the smallest admissible subset with respect to subset inclusion of D such that $\text{Cl}X \supseteq X$.
    (c) Show that the $(\text{Cl}X, \preceq)$ is a domain and that the inclusion map $\iota : \text{Cl}X \to D$ defined as $\iota \coloneqq \lambda x \in \text{Cl}X.\ x$ is continuous and strict.
    (d) Let E be another domain and suppose that f, g are strict and continuous functions from D to E which agree on X (i.e. $\forall x \in X. f(x) = g(x)$). Show that f and g agree on $\text{Cl}X$.
    (e) Let D be some domain and $X \subseteq D$. Define

$$Y \coloneqq X \cup \{\perp\} \cup \left\{ \bigsqcup_n x_n : \langle x_i \rangle \text{ is some chain in } X. \right\}$$

    Show that $Y \subseteq \text{Cl}X$. Then give an example where $Y \neq \text{Cl}X$.

[1] the LaTeXfor my name is `Daniel S\"a\"aw`.