Computer Science Tripos, Part II: Denotational Semantics
Supervision 3
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1. Is PCF Turing-complete? What conclusions can we draw about the relationship between computability and continuity? Which result from a previous supervision do you need to use to prove that the Halting problem is unsolvable?

2. What sets are denoted by

   (a) $\mathrm{PCF}_{\mathtt{nat}\to\mathtt{nat}}$

   (b) $[\![\mathtt{bool}]\!]$

   (c) $[\![\mathbb{N}]\!]$

   (d) $[\![\langle x \mapsto \mathtt{nat},\ y \mapsto \mathtt{bool},\ z \mapsto \mathtt{nat} \to \mathtt{bool}\rangle]\!]$

3. Are terms and expressions the same thing in PCF?

4. State the full, formal version of Proposition 5.4.1. Carry out the proof indicated in the lecture notes.

5. 2016, Paper 9, Question 5

6. What slide is the central result of this course presented on? What assumptions does its proof make?

7. 2009, Paper 9, Question 6

8. Define $\Omega_\tau \coloneqq \mathtt{fix}(\mathtt{fn}\ x : \tau.\ x)$. Show that $[\![\Omega_\tau]\!]$ is the least element of the domain $[\![\tau]\!]$. Then deduce that $[\![\Omega_{\tau\to\tau}]\!] = [\![\mathtt{fn}\ x : \tau.\ \Omega_\tau]\!]$.

9. The Fibonacci sequence can be computed in many ways, including the two Python functions below:

```python
def fib1(n):
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return fib1(n-1) + fib1(n-2)

def fib2(n):
    a = 1
    b = 0
    while n > 0:
        a, b = a+b, a
        n -= 1
    return b
```

   Define two PCF expressions computing the nth Fibonacci number in the two ways above. Show that they are contextually equivalent.

10. Exercise 7.4.1

11. Exercise 7.4.2

12. For each of the following statements, decide whether it is true or false, and justify why.

    (a) $\langle x \mapsto \mathtt{nat},\ y \mapsto \mathtt{nat}\rangle \vdash \mathtt{fn}\ y : \mathtt{nat}.\ \mathtt{succ}(y) \simeq_{\mathrm{ctx}} \mathtt{fn}\ x : \mathtt{nat}.\ \mathtt{succ}(x) : \mathtt{nat} \to \mathtt{nat}$

    (b) $x \simeq_{\mathrm{ctx}} x$

    (c) If $V \in \mathrm{PCF}_\tau$ is a value, $M \in \mathrm{PCF}_\tau$ and $V \simeq_{\mathrm{ctx}} M$, then $M \Downarrow_\tau V$.

    (d) $\bot_{\mathbb{N}} \lhd_{\mathtt{nat}} \mathtt{if}\ \mathtt{zero}(\mathtt{succ}(\mathtt{0}))\ \mathtt{then}\ x\ \mathtt{else}\ 0$

    (e) Since extending PCF with por is sufficient to get full abstraction, por is the only function in $\mathbb{B}_\bot \to \mathbb{B}_\bot \to \mathbb{B}_\bot$ that is not implementable in pure PCF.

13. 2021, Paper 9, Question 7

[1] the LaTeXfor my name is `Daniel S\"a\"aw`.